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On the ORBITS in which BODIES revolve, being acted upon by a CENTRIPETAL FORCE varying as any FUNCTION of the DISTANCE, when those ORBITS have TWO APSIDES. By the Rev. J. BRINKLEY, A. M. ANDREWS Professor of Astronomy in the University of Dublin.

THE investigation of orbits described by bodies acted upon by any centripetal force whatever is reduced by Sir Isaac Newton to the quadrature of curves (8 sect. lib. 1. Princip.) The quadrature of such curves as arise from the application of his method can only in few instances be completely accomplished. A portion of the area of any curve may be easily found by a converging series, but not to the whole area. To approximate to the whole area is in most cases very difficult; and hitherto the orbits have been investigated for very few laws of force. By the method here proposed it is shewn, that when the orbit has two apsidal, that whatever be the function of the distance which expresses the law of the centripetal force, the orbit may be determined by a series of sines of multiple arcs converging by the powers of the excentricity. From hence the angle between the

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apsides

apsides is immediately determined, which is one of the most interesting results of this method. For we have not only all that is determined in the last proposition of the 9th section of the *Principia*, but also the motion of the apsides for excentric orbits. The method in the 9th section gives only the limit of the motion of the apsides. It cannot be applied to find the motion in excentric orbits; which must in some measure be considered as a defect. The limit of the motion of the apsid is never required, for then the orbit is a circle; but the motion before it has arrived at its limit. The motion indeed approximates indefinitely to the limit, but this is not so evident from the method of Newton; we know from that *only* the limit, and nothing of its antecedent state. It must not be understood, that it is here intended to object to the truth of the reasoning in the 9th section; the ingenuity there shewn by the illustrious author is truly admirable, and is perhaps in no part of the *Principia* more striking. His penetrating mind, doubtless, saw at once the full force of that reasoning. It has, however, been a subject of difficulty to some. Walmsly, a very acute mathematician, found from the same data as in the 2 Cor. 45 Prop. a double motion of the apsides, and therefore consonant to the motion of the lunar apogee. He even has been followed by the ingenious Frisius, who, correcting, as he imagined, some defects in Walmsly's solution, found the same result as Walmsly. From which it would follow, that the mean motion of the lunar apogee could be found from the consideration

ration of a centripetal force, varying in a compounded ratio of the distance, and consequently that the 2d Cor. of the 45th Proposition of the 1st Book of the Principia is erroneous. The common error in the solutions of Walmisley and Frisius is hereafter pointed out.

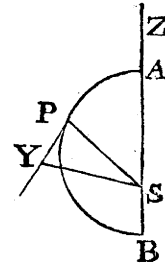
THE examples of the method here used are given in the 2d and 3d propositions. The 1st proposition proves, that an orbit, having two apfides, and described by a body impelled by a force varying inversely as the square of the distance, is an ellipse. This solution is probably more direct than any other of the same proposition, and has a connection with what follows. The series in the 2d and 3d propositions are not continued beyond the 2d and 3d powers of the excentricity. This was sufficient for the motion of the apfides, and there can be no difficulty in continuing the terms.

It seems hardly necessary to observe, that when the force varies as any function of the distance, an orbit can have only two different distances of the apfides from the centre, because it must be similar on each side of the apsid. The law of the force also readily shews, whether any orbit described by that law can have two apfides, by comparing it with the law of the centrifugal force, which always varies in the inverse triplicate ratio of the distance. Therefore if the force varies in any direct ratio, or any inverse ratio less than the triplicate, the orbits described may have two apfides.

Prop.

PROP. 1. The centripetal force varying inversely as the square of the distance to determine the orbit when it has two apfids.

SOLUTION. Let the greatest distance $SA = 1 + e$ and the least distance $SB = 1 - e$. Any distance $SP = x$, the perp. SY let fall on the tang. $= p$, the angle $ASP = a$, and the distance SZ from which a body acted upon by the centripetal force must fall to acquire the velocity at $A = z$.



THEN (Cor. 1. Prop. 1. Sect. 1. and Cor. 2. 40 Prop. 8 Sect. 1 Lib. Newtoni Prin.)

$$\overline{1+e^2} : \overline{1-e^2} :: \frac{1}{1-e} - \frac{1}{z} : \frac{1}{1+e} - \frac{1}{z} \text{ whence } z = 2$$

$$\text{and } \because p^2 : \overline{1+e^2} :: \frac{1}{1+e} - \frac{1}{2} : \frac{1}{x} - \frac{1}{2} \text{ or } p^2 = \frac{x}{2-x} \times \overline{1-e^2}$$

$$\text{Now } \dot{a} = \frac{SP \times SY}{PY \times SP} = \frac{-\dot{x}}{x \sqrt{\frac{x^2}{p^2} - 1}}$$

$$\therefore \dot{a} = \frac{-\dot{x}}{x \sqrt{\frac{x \times 2-x}{1-e^2} - 1}} = \frac{-\dot{x}}{x^2 \sqrt{\frac{2-x}{x} - \frac{1}{1-e^2} - \frac{1}{x^2}}} \quad \text{For finding the}$$

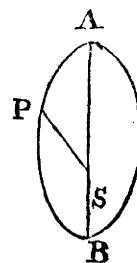
$$\text{fluent of this fluxion let } v = \frac{1}{1-e^2} - \frac{1}{x} \text{ and } \dot{a} = \frac{-\dot{v}}{\sqrt{e^2 - v^2}}$$

whence

Whence a = the arc, the cofine of which is $\frac{v}{e} \times \overline{1-e^2}$ and rad.
 unity = the arc, the cofine of which is $\frac{1}{e} - \frac{1-e^2}{e x}$. This fluent
 requires no correction, because when $a = 0$, $\frac{v}{e} \times \overline{1-e^2} = 1$. And
 because $cs, a = \frac{1}{e} - \frac{1-e^2}{e x}$, $\therefore x = \frac{1-e^2}{1-e cs, a}$, which is the well
 known equation of the ellipse, in which x is the distance from
 the focus and a the true anomaly QEI .

FOR the subsequent propositions it is necessary to solve the
 following problem :

PROBLEM. Let the greatest distance SA from
 the focus S of the ellipse $APB = 1+e$, the least
 distance $SB = 1-e$, any other distance $SP = 1+y$,
 and the angle $ASP = a$, it is required to ex-
 press the different integral powers of y by a series
 of cofines of multiple arcs and powers of the ex-
 centricity.



SOLUTION. The equation of the ellipse gives

$$1+y = \frac{1-e^2}{1-e cs, a} \therefore y = e \times \frac{cs, a-e}{1-e cs, a}$$

or

now $\frac{1}{1 - e \cos a} = 1 + e \cos a + e^2 \cos^2 a + e^3 \cos^3 a + \&c.$

$\therefore y = -e^2 + e - e^3 \cos a + e^2 \cos^2 a + e^3 \cos^3 a + \&c. =$

$-\frac{e^2}{2} + \frac{e - e^3}{4} \cos a + \frac{1}{2} e^2 \cos^2 a + \frac{1}{4} e^3 \cos^3 a + \&c.$ This series

might be continued ad libitum from the relation of the terms to each other, which relation may be found by a method well known, but the above is sufficient for the present purpose. Hence

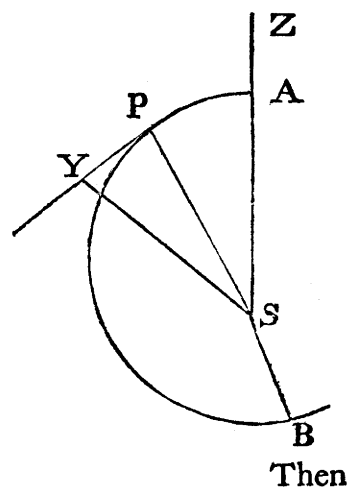
$y^2 = \frac{e^2}{2} - \frac{e^3}{2} \cos a + \frac{e^2}{2} \cos^2 a + \frac{e^3}{2} \cos^3 a + \&c.$

$y^3 = \frac{3}{8} e^3 \cos a + \frac{1}{4} e^3 \cos^3 a + \&c.$

&c. &c.

PROP. 2. The centripetal force varying partly in the inverse duplicate ratio, and partly in the direct simple ratio of the distance to determine the orbit described, when that orbit has two apfides. (Newtoni Prin. Sect. 9. Prop. 45. Cor. 2.)

SOLUTION. Let, as before, the greatest distance $SA = 1 + e$, the least dist. $SB = 1 - e$, the angle $ASP = A$, the distance $SP = x$, the perp. $SY = p$, and let the force at P be expressed by $\frac{1}{x^2} - cx$. Let also SZ be the height from which a body must fall to acquire the velocity at A .



THEN (39 and 40 Prop. 1. Lib. Prin.) $\frac{1}{x} - \frac{1}{z} \rightarrow \frac{1}{2} c \times \overline{z^2 - x^2}$
 may express the square of the velocity at any point P \therefore (1 Cor.
 1 Prop. 1 Lib.)

$$\frac{1}{1+e} - \frac{1}{z} - \frac{1}{2} c \times \overline{z^2 - 1+e}^2 : \frac{1}{1-e} - \frac{1}{z} - \frac{1}{2} c \times \overline{z^2 - 1-e}^2 :: \overline{1-e}^2 : \overline{1+e}^2$$

Whence $\frac{1}{z} - \frac{1}{2} c z^2 = \frac{1}{2} - \frac{1}{2} c \times \overline{2 + 2e^2}$

Consequently $\frac{1}{x} - \frac{1}{2} - \frac{1}{2} c \times \overline{2 + 2e^2 - x^2} : \frac{1}{1+e} - \frac{1}{2} - \frac{1}{2} c \times \overline{2 + 2e^2 - 1+e^2} :: \overline{1+e^2}^2 : p^2$

or $\frac{x^2}{p^2} = \frac{2x - x^2 - c x^2 \times 2 + 2e^2 - x^2}{1 - e^2 - c \times 1 - e^2}$

And $\therefore \dot{A} = \frac{\dot{x}}{\sqrt{\frac{x^2}{p^2} - 1}} = \frac{\dot{x}}{\sqrt{\frac{2x - x^2 - c x^2 \times 2 + 2e^2 - x^2}{1 - e^2 - c \times 1 - e^2} - 1}}$

THE fluent of this quantity may be approximated by help of
 an ellipse, the greatest and least focal distances of which are
 $1+e$ and $1-e$, and a the angle included between the greatest
 E c distance

distance $1+e$ and any distance x . In this ellipse (see the preceding prop.)

$$\dot{a} = \frac{-x}{x \sqrt{\frac{2x-x^2}{1-e^2}} - 1}$$

$$\therefore \dot{A} : \dot{a} :: \sqrt{\frac{1-c \times 1-e^2}{1-c \times \frac{2x-x^2}{1-e^2} - 1}} : 1 \quad \text{Now because two}$$

roots of the equation $P=0$, viz. $1+e$ and $1-e$ are the roots of the equation $Q=0$, it follows that Q must be a divisor of P ;

accordingly we find $\frac{P}{Q} = \frac{x+1}{x} - e^2 = \frac{2+y}{2} - e^2$, putting $1+y=x$

$$\text{Hence } \dot{A} = \dot{a} \sqrt{\frac{1-c}{1-4c+ce^2}} \times \sqrt{\frac{1}{1-c \cdot \frac{2y+y^2}{1-4c+ce^2}}}$$

$= \dot{a} \sqrt{\frac{1-c}{1-4c+ce^2}} \times \left(1 + \frac{c}{2B} \cdot \frac{2y+y^2}{2y+y^2} + \frac{3c^2 y^2}{2B^2} + \&c. \right)$ regarding only the 2d power of the eccentricity, which is sufficient for the purpose for which the proposition was designed. Next substituting for $y, y^2, \&c.$ the values found in the preceding problem, and taking the fluent, we have

$$A = a \sqrt{\frac{1-c}{1-4c+ce^2}} \times \left(1 - \frac{\frac{ce^2}{4} - ce^2 s, a - \frac{3}{8} ce^2 s, 2a}{1-4c+ce^2} + \right.$$

$$\left. \frac{\frac{3}{4} e^2 c^2 + \frac{3}{4} e^2 c^2 s, 2a}{1-4c+ce^2} \right)^2, \&c.$$

where

where a is an arc the cosine of which is $\frac{1}{e} - \frac{1-e^2}{e x}$ rad. being $\frac{1}{e}$.

Hence for every dist. S P we have the angle A S P, and consequently the orbit is determined. When the body comes to the lower apsid, $a = 180^\circ$ \therefore the angle between the apsides $= 180^\circ$.

$$\sqrt{\frac{1-c}{1-4c+c^2}} \times 1 - \frac{c e^2}{4 \cdot 1-4c+e^2 c} + \frac{3 e^2 c^2}{4 \cdot 1-4c+e^2 c^2}, \text{ \&c.}$$

The limit of this quantity is $180^\circ \int \frac{1-c}{1-4c} (2 \text{ Cor. 45 Prop.})$

THIS proposition is applicable to the lunar orbit. The limit of the result is the same as found by Sir Isaac Newton. Some authors have conceived Newton's conclusion erroneous, and with the same law of force have found the motion of the apsides twice as great. Walmfly, particularly, has imagined, that the principles of the 9th section give the true angle between the apsides only when the force varies according to a simple law of the distance. In his tract, "De Inæqual. Lunæ," he finds the motion of the apsides, by computing the time a body takes in acceding towards the centre a space equal to twice the excentricity, when impelled by a force which is the difference of the centrifugal and centripetal forces. This time he compares with half the periodic time, and thence

deduces the motion of the apfides to be twice as great as by the principles of the 9th section. Frifius, observing that Walmsly in his method had omitted the disturbing tangential force as of no effect in its mean quantity, endeavours to correct his solution by using the mean velocity of the moon in octants, and her mean periodic time as affected by the tangential force. He then finds the result the same as Walmsly. But upon examining his method, it will be seen it does not differ essentially from Walmsly's. Increasing the velocity, and decreasing the periodic time, does not affect the angle between the apfides. The motions of the apfides in orbits little excentric, as the above and next proposition shew, almost entirely depend upon the variation of centripetal force. The variation of the force in Walmsly's and Frifius's methods will, upon examination, be found to be precisely the same as Newton's, and therefore the motion of the apfides ought to be the same. The errors in the processes of Walmsly and Frifius are exactly alike. The space used by them for finding the time is only an approximation to the excentricity. At the end of the space the velocity is evanescent, and from that circumstance the fluent of the fluxional expression of the time must be erroneous.

TAKING $e=.055$, as in the lunar orbit, the angle between the apfides will differ about 4 seconds from the limit, consequently
the

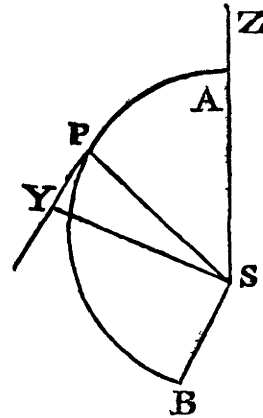
the error in the mean motion of the apsides of the lunar orbit, by neglecting the excentricity, is only 8 seconds in a revolution.

IN the lunar orbit referred to the ecliptic, the perturbing force in the direction of the radius vector is expressed by a function of that radius vector, and of the angular distance of the moon from the sun; and the perturbing force in a direction perp. to the radius vector, is expressed by another function of the same quantities. The former force in its mean quantity is expressed by a function of the radius vector only. The mean quantity of the latter = 0. It has therefore been often imagined, that the mean motion of the lunar apogee might be investigated, by considering the moon acted upon by a centripetal force, expressed by a function of the distance only. The arguments for this opinion are certainly plausible, but have by no means the weight of demonstration. The result shews, that such an opinion rests upon no solid foundation. It does not appear to be possible to investigate the mean motion of the lunar apsides, except from the general expressions of the forces in direction of the radius vector and in the direction perp. thereto.

PROP.

PROP. 3. The centripetal varying as the $n-1$ power of the distance to determine the orbit described when it has two apsides.

SOLUTION. Let $SA = 1 + e$ and $SB = 1 - e$. Any distance $SP = x$, $SY = p$ and the $\angle ASP = A$. Let also ZA be the space thro' which a body must fall to acquire the velocity at A , and let $ZS = z$



Then (40 Prop. Cor. 2. and 1 Prop. Cor. 1. Prin. Newtoni)

$$x^n - \sqrt[n]{1+e} : x^n - \sqrt[n]{1-e} :: \sqrt[n]{1-e}^2 : \sqrt[n]{1+e}^2$$

or compon. $z^n - \overline{1+e}^n : \overline{1+e}^n - \overline{1-e}^n :: \overline{1-e}^2 : \overline{1+e}^2 - \overline{1-e}^2$

$$\text{Whence } x_n = \frac{1 - e^{n+2} - 1 - e^{n+2}}{4e} = \frac{n+2}{2} + \frac{n+2 \cdot n+1 \cdot n}{2 \cdot 2 \cdot 3} e^2 +$$

$$\frac{n+2 \cdot n+1 \cdot n \cdot n-1 \cdot n-2}{2 \cdot 2 \cdot 3 \cdot 4 \cdot 5} e^2 \quad \&c.$$

and because $z^n - x^n : z^n - \overline{1 - e}^n :: \overline{1 + e}^n : p^2$

we

$$\text{we have } \dot{A} = \frac{-\dot{x}}{x \sqrt{x^2 - 1}} = \frac{-\dot{x}}{x \sqrt{\frac{x^n x^2 - x^{n+2}}{x^n - 1 + e^n \times 1 + e^2}} - 1}$$

For finding the fluent of this fluxion we may use the ellipse as in the last prop.

$$\text{and } \dot{A} : \dot{a} :: \frac{1}{\sqrt{\frac{x^n x^2 - x^{n+2}}{x^n - 1 + e^n \times 1 + e^2}} - 1} : \frac{1}{\sqrt{\frac{2x - x^2}{1 - e^2}} - 1}$$

$$\text{or } \dot{A} = \dot{a} \sqrt{\frac{x^n - 1 + e^n \times 1 + e^2}{1 - e^2}} (R) \times \frac{1}{\sqrt{\frac{x^{n+2} - x^n x^2 + x^n - 1 + e \times 1 + e^2 (P)}{x^2 - 2x + 1 - e^2}} (Q)}$$

Now $1 + e$ and $1 - e$ are values of x in the equation $P = 0$, and the same quantities are the values of x in the equation $Q = 0$ \therefore Q must be a divisor of P whatever be the value of n . To accomplish this division let $x = 1 + y$

$$\text{and then } P = \left. \begin{aligned} & -x^n \times \overline{1 + y}^2 + \overline{1 + y}^{n+2} \\ & + x^n \times \overline{1 + e}^2 - \overline{1 + e}^{n+2} \end{aligned} \right\} =$$

$$-x^n \times \overline{2y - 2e} - x^n \times \overline{y^2 - e^2} + \overline{n + 2. y - e} + \frac{\overline{n + 2. n + 1}}{1.2} \overline{y^2 - e^2} +$$

$$\frac{\overline{n + 2. n + 1. n}}{1.2.3} y^3 - e^3 + \&c.$$

$$= -z^n \times \frac{\overline{1} \cdot \overline{2} \cdot \overline{3} \cdots \overline{n}}{\overline{1} \cdot \overline{2}} + \frac{\overline{n+2} \cdot \overline{n+1}}{\overline{1} \cdot \overline{2}} \frac{\overline{1} \cdot \overline{2} \cdot \overline{3} \cdots \overline{n}}{\overline{1} \cdot \overline{2}} y^2 - e^2 + \frac{\overline{n+2} \cdot \overline{n+1} \cdot \overline{n}}{\overline{2} \cdot \overline{3}} \frac{\overline{1} \cdot \overline{2} \cdot \overline{3} \cdots \overline{n}}{\overline{2} \cdot \overline{3}} y^3 - e^2 y +$$

$$\frac{\overline{n+2} \cdot \overline{n+1} \cdot \overline{n} \cdot \overline{n-1}}{\overline{2} \cdot \overline{3} \cdot \overline{4}} \frac{\overline{1} \cdot \overline{2} \cdot \overline{3} \cdots \overline{n}}{\overline{2} \cdot \overline{3} \cdot \overline{4}} y^4 - e^2 y^2 + \frac{\overline{n+2} \cdot \overline{n+1} \cdot \overline{n} \cdot \overline{n-2}}{\overline{2} \cdot \overline{3} \cdot \overline{4} \cdot \overline{5}} \frac{\overline{1} \cdot \overline{2} \cdot \overline{3} \cdots \overline{n}}{\overline{2} \cdot \overline{3} \cdot \overline{4} \cdot \overline{5}} y^5 - e^2 y^3 + \&c. \text{ each term}$$

of this quantity is divisible by $y^2 - e^2 = Q$, because $y^{2m} - e^{2m}$ (m being a whole number) is always divisible by $y^2 - e^2$

$$\therefore \frac{P}{Q} = -z^n + \frac{\overline{n+2} \cdot \overline{n+1}}{\overline{1} \cdot \overline{2}} + \frac{\overline{n+2} \cdot \overline{n+1} \cdot \overline{n}}{\overline{2} \cdot \overline{3}} y + \frac{\overline{n+2} \cdot \overline{n+1} \cdot \overline{n} \cdot \overline{n-1}}{\overline{2} \cdot \overline{3} \cdot \overline{4}} y^2 + e^2 +$$

$$+ \frac{\overline{n+2} \cdot \overline{n+1} \cdot \overline{n} \cdot \overline{n-2}}{\overline{2} \cdot \overline{3} \cdot \overline{4} \cdot \overline{5}} y^3 + e^2 y + \frac{\overline{n+2} \cdot \overline{n+1} \cdot \overline{n} \cdot \overline{n-3}}{\overline{2} \cdot \overline{3} \cdot \overline{4} \cdot \overline{5} \cdot \overline{6}} y^4 + e^2 y^2 + e^4 + \&c. =$$

$$\frac{\overline{n+2} \cdot \overline{n}}{\overline{1} \cdot \overline{2}} \times 1 + \frac{\overline{n+1} \cdot \overline{n-3}}{\overline{3} \cdot \overline{4}} e^2 + \frac{\overline{n+1} \cdot \overline{n-1} \cdot \overline{n-2} \cdot \overline{n-4}}{\overline{3} \cdot \overline{4} \cdot \overline{5} \cdot \overline{6}} e^4 + \frac{\overline{n+1}}{\overline{3}} y +$$

$$\frac{\overline{n+1} \cdot \overline{n-1}}{\overline{3} \cdot \overline{4}} y^2 + \frac{\overline{n+1} \cdot \overline{n-1} \cdot \overline{n-2}}{\overline{3} \cdot \overline{4} \cdot \overline{5}} y^3 + e^2 y + \frac{\overline{n+1} \cdot \overline{n-1} \cdot \overline{n-2} \cdot \overline{n-3}}{\overline{3} \cdot \overline{4} \cdot \overline{5} \cdot \overline{6}} y^4 + e^2 y^2 + \&c.$$

$$\text{Also } \frac{I}{R} = \frac{2}{n} \times 1 - \frac{\overline{n+1} \cdot \overline{n-4}}{\overline{2} \cdot \overline{3}} e^2 + \&c. \text{ It is evident from the}$$

process for the values of R and $\frac{P}{Q}$ that only the even powers of the eccentricity can enter into those values.

Substituting

Substituting these values, $\dot{A} = \dot{a} \sqrt{\frac{RQ}{P}} =$

$$\sqrt{\frac{\dot{a}}{n+2} \times \left(1 - \frac{n+1}{3 \cdot 4} e^2 + \frac{n+1}{3} y + \frac{n+1 \cdot n-1}{3 \cdot 4} y^2 + \frac{n+1 \cdot n-1 \cdot n-2}{3 \cdot 4 \cdot 5} y^3 + e^2 y - \frac{n+1^2 \cdot n-4}{2 \cdot 3 \cdot 3} e^2 y \text{ \&c.} \right) - \frac{1}{2}}$$

$$= \frac{\dot{a}}{n+2} \times \left\{ 1 + \frac{n+1}{2 \cdot 3 \cdot 4} e^2 - \frac{n+1}{2 \cdot 3} y + \frac{n+1}{3 \cdot 4} y^2 + \frac{n+1^2 \cdot n-1}{3 \cdot 4 \cdot 4} - \frac{n+1 \cdot n-1 \cdot n-2}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{5 \cdot n+1^3}{3 \cdot 4 \cdot 4 \cdot 9} \right\} y^2 + \frac{n+1^2 \cdot n-4}{2 \cdot 2 \cdot 3 \cdot 3} - \frac{n+1^2 \cdot n-5}{3 \cdot 4 \cdot 4} \left\} e^2 y \text{ \&c.}$$

Next substituting for $y, y_1, y_2, \text{ \&c.}$ their values found in the foregoing problem, and taking the fluent we have

$$A = \frac{a}{n+2} \times \left\{ 1 + \frac{n+1}{2 \cdot 3} \times \left(\frac{n-2}{4} e^2 - e s, a + \frac{17 \cdot n+1 \cdot n-1}{8 \cdot 8} - \frac{5 \cdot n+1^2}{3 \cdot 4 \cdot 4} - \frac{11 \cdot n-1 \cdot n-2}{4 \cdot 5 \cdot 8} \right) - \frac{e^2 s, 2a +}{2 \cdot 4} \right\}$$

$$- \frac{\frac{n+1 \cdot n-1}{2 \cdot 3 \cdot 3 \cdot 4 \cdot 4}}{n-1 \cdot n-2} \left\{ e^3 s, 3a, \text{ \&c.} \right\} \text{ where } a \text{ is an arc, the cosine of}$$

which is $\frac{1}{e} - \frac{1-e^2}{e x}$ and rad. 1. This is the general equation

for any distance and angle. When the body arrives at the lower

apfid, $a = 180^\circ$ and $s, a; s, 2a$ &c. $= 0 \therefore$ the angle between the apfides =

$$\frac{180^\circ}{n+2} \times 1 + \frac{n+1 \cdot n-2}{2 \cdot 3 \cdot 4} e^2, \text{ \&c. in which only the even}$$

powers of the eccentricity can enter. It is evident from this expression that the motion of the apfides will be always affected by the eccentricity of the orbit, unless either $n+1=0$ or $n-2=0$ or that the force varies either in the inverse duplicate ratio of the distance or in the direct simple ratio. We know from other principles that in these two laws the eccentricity does not affect the angle between the apfides, but it is a very remarkable circumstance that this takes place for no other law.

THE above propositions point out how the orbit may be found whatever be the function of the distance expressing the centripetal force. For x will be always found a function of e , and therefore p will always be a function of x and e , consequently $\frac{x^2}{p^2} - 1$ will be a function of x and e , or substituting for x , $1+y$, $\frac{x^2}{p^2} - 1$ will be a function of y and e . And two roots of the equation

equation $\frac{(1+y)^2}{p^2} - 1 = 0$ are $\pm e \therefore y^2 - e^2$ will be always a divisor of $\frac{x^2}{p^2} - 1$ or $\frac{(1+y)^2}{p^2} - 1$. Whence in every case we may proceed as above and shall always have the equation of the orbit in a series of sines of multiple arcs converging by the powers of the excentricity.